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ON UNSTEADY BOUNDARY LAYER WITH SELF-INDUCED PRESSURE

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Asymtotic equations which define the unsteady processes in a boundary layer with self-induced pressure are derived. The pressure gradient is not determined by the solution of the external flow, but is assumed to be defined by the increase of the displacement thickness of flow filaments situated close to the body surface. Principal terms and terms of second order of smallness are retained in asymptotic sequences. A solution that satisfies the linearized system of equations is derived for the principal expansion terms. The relation between the considered nonlinear phenomena and stability of the boundary layer is indicated.

Experiments carried out in 1946 had disclosed a remarkable phenomenon: separation of the supersonic boundary layer is induced by a shock wave at some distance upstream from it [1, 2]. Its qualitative explanation was suggested in [3]. Prior to that the idea that pressure is induced by the boundary layer itself owing to a sudden increase of the flow filament thickness in the thin boundary layer was proposed for the quantitative description of that phenomenon within the confines of the linear theory [4]. The complete theory of perturbations is of the nonlinear kind; it was developed in [5-9], and made possible the analysis of the velocity field structure in the neighborhood of the separation point and downstream of it.

1. Asymptotic Equations. We shall distinguish three regions of different properties of free interaction between the unsteady boundary layer and the external flow [5-9]. In the upper region I the effects of viscosity and thermal conductivity are small and the flow is vortex-free. In the intermediate region II the effect of dissipative factors can also be neglected but the flow field is no longer vortex-free. Viscosity is the determining factor of the flow pattern in region III of the narrow boundary layer, while the effect of heat conduction is secondary, since at low velocities the compressibility of gas is virtually absent, if its temperature varies in a fairly narrow range.

We denote time by t, the Cartesian coordinates by x and y, the velocity vector components along these axes by v_x and v_y, the density by ρ, the pressure by p, and the first coefficient of viscosity by λ_1. The parameters of the unperturbed gas in the steady state are denoted by subscript ∞. We assume for simplicity that the gas flows along the plate at velocity U_∞, and that the Mach number M_∞ differs from unity by a finite quantity. We introduce the small parameter $\varepsilon = \mathrm{Re}_1^{-1/8}$, and calculate the Reynolds number Re_1 using the first viscosity coefficient λ_1 and distance L from the plate leading edge.

We begin by analyzing the external region I in which the velocity field is vortex-free. We set here

$$t = \frac{L}{U_\infty}(t_0 + \varepsilon^2 t_1), \quad x = L(1 + \varepsilon^3 x_1), \quad y = \varepsilon^3 L y_1 \tag{1.1}$$

and expand the unknown functions into asymptotic series

$$
\begin{aligned}
v_x &= U_\infty [1 + \varepsilon^2 u_{11}(t_1, x_1, y_1) + \varepsilon^3 u_{12}(t_1, x_1, y_1) + \ldots] \\
v_y &= U_\infty [\varepsilon^2 v_{11}(t_1, x_1, y_1) + \varepsilon^3 v_{12}(i_1, x_1, y_1) + \ldots] \\
\rho &= \rho_\infty [1 + \varepsilon^2 \rho_{11}(t_1, x_1, y_1) + \varepsilon^3 \rho_{12}(t_1, x_1, y_1) + \ldots] \\
p &= p_\infty + \rho_\infty U_\infty^2 [\varepsilon^2 p_{11}(t_1, x_1, y_1) + \varepsilon^3 p_{12}(t_1, x_1, y_1) + \ldots]
\end{aligned} \tag{1.2}
$$

We substitute formulas (1.1) and expansions (1.2) into the system of Navier-Stokes equations and collect terms of like powers of ε. For the first approximation functions we obtain

$$
\begin{aligned}
&\frac{\partial u_{11}}{\partial x_1} + \frac{\partial \rho_{11}}{\partial x_1} + \frac{\partial v_{11}}{\partial y_1} = 0, \quad \frac{\partial u_{11}}{\partial x_1} + \frac{\partial p_{11}}{\partial x_1} = 0 \\
&\frac{\partial v_{11}}{\partial x_1} + \frac{\partial p_{11}}{\partial y_1} = 0, \quad M_\infty^2 \frac{\partial p_{11}}{\partial x_1} - \frac{\partial \rho_{11}}{\partial x_1} = 0
\end{aligned} \tag{1.3}
$$

It is important that all equations of system (1.3) do not contain time derivatives. This means that the external inviscid flow is inert and adjusts itself instantly to perturbations which, as shown below, originate in region III of the boundary layer. Second approximation functions satisfy the nonhomogeneous system of linear equations

$$
\begin{aligned}
&\frac{\partial u_{12}}{\partial x_1} + \frac{\partial \rho_{12}}{\partial x_1} + \frac{\partial v_{12}}{\partial y_1} = -\frac{\partial \rho_{11}}{\partial t_1}, \quad \frac{\partial u_{12}}{\partial x_1} + \frac{\partial p_{12}}{\partial x_1} = -\frac{\partial u_{11}}{\partial t_1} \\
&\frac{\partial v_{12}}{\partial x_1} + \frac{\partial p_{12}}{\partial y_1} = -\frac{\partial v_{11}}{\partial t_1}, \quad M_\infty^2 \frac{\partial p_{12}}{\partial x_1} - \frac{\partial \rho_{12}}{\partial x_1} = 0
\end{aligned} \tag{1.4}
$$

Its corresponding homogeneous system conforms to system (1.3). Time appears only in the right-hand sides of Eqs. (1.4), and in their solutions it is, also, contained as a parameter.

We carry out a partial integration of the system of Eqs. (1.3) and (1.4) on condition that all unknown functions tend to zero when $x_1 \to -\infty$ with $y_1 = \text{const}$ and $y_1 \to +\infty$ with $x_1 = \text{const}$. The first of these systems yields

$$
\begin{aligned}
&u_{11} + p_{11} = 0, \quad \rho_{11} - M_\infty^2 p_{11} = 0 \\
&\frac{\partial v_{11}}{\partial y_1} + (M_\infty^2 - 1)\frac{\partial p_{11}}{\partial x_1} = 0, \quad \frac{\partial v_{11}}{\partial x_1} + \frac{\partial p_{11}}{\partial y_1} = 0
\end{aligned}
$$

It is readily seen that in the case of supersonic flow at $M_\infty > 1$ each of the functions p_{11} and v_{11} satisfies the wave equation. From this we have the relation

$$
p_{11}(t_1, x_1, 0) = (M_\infty^2 - 1)^{-1/2} v_{11}(t_1, x_1, 0) \tag{1.5}
$$

between the perturbed pressure and the vertical component of the velocity vector when $y_1 = 0$. For a subsonic oncoming flow we have Neumann's problem for the Laplace equation whose solution is sought in the half-plane $y_1 > 0$. Hence when $M_\infty < 1$ we obtain

$$
p_{11}(t_1, x_1, 0) = -\frac{1}{\pi}(1 - M_\infty^2)^{-1/2} \int_{-\infty}^{+\infty} \frac{v_{11}(t_1, X_1, 0)}{x_1 - X_1'} \, dX_1 \tag{1.6}
$$

The partial integration of the system of equations for the second approximation functions yields

$$\frac{\partial u_{12}}{\partial x_1}+\frac{\partial p_{12}}{\partial x_1}=\frac{\partial p_{11}}{\partial t_1},\quad \rho_{12}-M_\infty{}^2 p_{12}=0$$
$$\frac{\partial v_{12}}{\partial y_1}+(M_\infty{}^2-1)\frac{\partial p_{12}}{\partial x_1}=-(M_\infty{}^2+1)\frac{\partial p_{11}}{\partial t_1}$$
$$\frac{\partial v_{12}}{\partial x_1}+\frac{\partial p_{12}}{\partial y_1}=-\frac{\partial v_{11}}{\partial t_1}$$

If the flow is supersonic, both functions p_{12} and v_{12} are solutions of the wave equation containing a right-hand side. When $y_1 = 0$ these are related by formula

$$p_{12}(t_1, x_1, 0) = (M_\infty{}^2-1)^{-1/2}\Big[v_{12}(t_1,x_1,0) - (M_\infty{}^2-1)^{-1}\int\limits_{-\infty}^{x_1}\frac{\partial v_{11}(t_1,X_1,0)}{\partial t_1}dX_1\Big] \qquad (1.7)$$

In the case of subsonic flow Neumann's problem may be formulated for the Poisson equation whose solution is to be determined in the upper half-plane $y_1 > 0$. Simple transformations show that then

$$p_{12}(t_1,x_1,0) = -\frac{1}{\pi}(1-M_\infty{}^2)^{-1/2}\int\limits_{-\infty}^{+\infty}\Big[\frac{v_{12}(t_1,X_1,0)}{x_1-X_1} - \frac{\partial v_{11}(t_1,X_1,0)}{\partial t_1}\ln\frac{1}{|x_1-X_1|}\Big]dX_1 + \qquad (1.8)$$
$$\frac{1}{\pi}\frac{M_\infty{}^2}{1-M_\infty{}^2}\int\limits_0^\infty\int\limits_{-\infty}^{+\infty}\frac{\partial^2 p_{11}(t_1,X_1,Y_1)}{\partial t_1\partial X_1'}\Big\{-\ln\frac{1}{\sqrt{(x_1-X_1)^2+Y_1^2}} +$$
$$\frac{1}{\pi}\Big[\arccos\Big(\frac{-x_1+X_1}{\sqrt{(x_1-X_1)^2+Y_1^2}}\Big)\ln\sqrt{(x_1-X_1)^2+Y_1^2} +$$
$$\arccos\Big(\frac{-x_1-X_1}{\sqrt{(x_1+X_1)^2+Y_1^2}}\Big)\ln\sqrt{(x_1+X_1)^2+Y_1^2}\Big]\Big\}dX_1dY_1$$

Let us pass to the investigation of the intermediate region II which contains the basic part of the boundary layer. Although it is possible to neglect in that region viscous stresses and the heat flux, it is necessary to take into account even in the first approximation the vorticity of the flow. The time and coordinate scales are specified by formulas

$$t = \frac{L}{U_\infty}(t_0+\varepsilon^2 t_2),\quad x = L(1+\varepsilon^3 x_2),\quad y=\varepsilon^4 L y_2 \qquad (1.9)$$

For the gas parameters the following expansions are valid:

$$v_x = U_\infty[U_0(y_2)+\varepsilon u_{21}(t_2,x_2,y_2)+\varepsilon^2 u_{22}(t_2,x_2,y_2)+\ldots] \qquad (1.10)$$
$$v_y = U_\infty[\varepsilon^2 v_{21}(t_2,x_2,y_2)+\varepsilon^3 v_{22}(t_2,x_2,y_2)+\ldots]$$
$$\rho = \rho_\infty[R_0(y_2)+\varepsilon\rho_{21}(t_2,x_2,y_2)+\varepsilon^2\rho_{22}(t_2,x_2,y_2)+\ldots]$$
$$p = p_\infty+\rho_\infty U_\infty{}^2[\varepsilon^2 p_{21}(t_2,x_2,y_2)+\varepsilon^3 p_{22}(t_2,x_2,y_2)+\ldots]$$

Comparison of formulas (1.1) and (1.9) shows, first of all, that $t_1 = t_2$ and $x_1 = x_2$, but $y_1 \neq y_2$. In region II the Cartesian coordinates are of different scales, because the characteristic length in the transverse direction has been selected equal to the thickness of the unperturbed boundary layer of the plate. Structure of the latter is determined by the Blasius solution [10], merging with which at $x_1 \to -\infty$ makes it possible to establish the form of functions $U_0(y_2)$ and $R_0(y_2)$.

By substituting formulas (1.9) and expansions (1.10) into the system of Navier-Stokes equations for the principal terms we obtain

$$R_0 \frac{\partial u_{21}}{\partial x_2} + U_0 \frac{\partial \rho_{21}}{\partial x_2} + R_0 \frac{\partial v_{21}}{\partial y_2} + v_{21} \frac{\partial R_0}{d y_2} = 0 \tag{1.11}$$

$$U_0 \frac{\partial u_{21}}{\partial x_2} + \frac{d U_0}{d y_2} = 0, \quad \frac{\partial p_{21}}{\partial y_2} = 0 \quad U_0 \frac{\partial \rho_{21}}{\partial x_2} + v_{21} \frac{d R_0}{d y_2} = 0$$

Here again all equations of system (1.11) do not contain time derivatives. In the first approximation oscillations in the basic part of the boundary layer are instantly transmitted from point to point. The essentially unstable pattern of flow can only occur in the thin boundary layer. The system of equations for the correction factors in expansions (1.10) is nonhomogeneous

$$R_0 \frac{\partial u_{22}}{\partial x_2} + U_0 \frac{\partial \rho_{22}}{\partial x_2} + R_0 \frac{\partial v_{22}}{\partial y_2} + v_{22} \frac{d R_0}{d y_2} = \tag{1.12}$$

$$- \frac{\partial \rho_{21}}{\partial t_2} - \frac{\partial \rho_{21} u_{21}}{\partial x_2} - \frac{\partial \rho_{21} v_{21}}{\partial y_2}$$

$$R_0 U_0 \frac{\partial u_{22}}{\partial x_2} + R_0 v_{22} \frac{d U_0}{d y_2} = - \frac{\partial p_{21}}{\partial x_2} - R_0 \frac{\partial u_{21}}{\partial t_2} -$$

$$R_0 u_{21} \frac{\partial u_{21}}{\partial x_2} - R_0 v_{21} \frac{\partial u_{21}}{\partial y_2} \quad \frac{\partial p_{22}}{\partial y_2} = - R_0 U_0 \frac{\partial v_{21}}{\partial x_2}$$

$$U_0 \frac{\partial \rho_{22}}{\partial x_2} + v_{22} \frac{d R_0}{d y_2} = - \frac{\partial \rho_{21}}{\partial t_2} - u_{21} \frac{\partial \rho_{21}}{\partial x_2} -$$

$$v_{21} \frac{\partial \rho_{21}}{\partial y_2} - R_0 U_0 M_\infty^2 \frac{\partial p_{21}}{\partial x_2}$$

The form of its corresponding homogeneous system is the same as that of system (1.11), although it is linear. Since time appears only in terms in the right-hand sides of Eqs. (1.12), hence it will appear in its solutions as a parameter. The parametric dependence on time is, thus, a distinctive feature of expansions which specify the perturbed flow field in both the upper region I and in the intermediate region II.

To integrate the last two systems of equations we specify that perturbations in region II must die out at infinity upstream of the flow. For the principal terms we have the following explicit formulas:

$$u_{21} = A_1(t_2, x_2) \frac{d U_0}{d y_2}, \quad v_{21} = - \frac{\partial A_1}{\partial x_2} U_0(y_2) \tag{1.13}$$

$$\rho_{21} = A_1(t_2, x_2) \frac{d R_0}{d y_2}, \quad p_{21} = p_{21}(t_2, x_2)$$

The arbitrary function $A_1(t_2, x_2)$ satisfies the condition $A_1 \to 0$ when $x_2 \to -\infty$ and $t_2 = \text{const}$. The meaning of this simple solution is that the streamlines in the boundary layer become displaced; the instantaneous displacement is determined by substituting $y_2 + \varepsilon A_1(t_2, x_2)$ for y_2 in the Blasius solution.

The system of Eqs. (1.12) can be partially integrated. Taking into account the explicit form of solution for the first approximation, we obtain

$$p_{22} = p_{22}(t_2, x_2, 0) + \left(y_2 - \int_0^{y_2} \frac{M_\infty^2 - M_0^2}{M_\infty^2} dY_2\right) \frac{\partial^2 A_1}{\partial x_2^2}, \tag{1.14}$$

$$M_0^2 = M_\infty^2 R_0(y_2) U_0^2(y_2)$$

$$v_{22} = -(M_\infty^2 - 1) y_2 U_0 \frac{\partial p_{21}}{\partial x_2} - M_\infty^2 U_0 \frac{\partial p_{21}}{\partial x_2} \int_{y_2}^{\infty} \left(\frac{1}{M_0^2} - \frac{1}{M_\infty^2}\right) dY_2 -$$

$$\frac{\partial A_1}{\partial t_2} - A_1 \frac{\partial A_1}{\partial x_2} \frac{dU_0}{dy_2} - U_0 \frac{\partial A_2(t_2, x_2)}{\partial x_2}$$

$$\frac{\partial u_{22}}{\partial x_2} + \frac{\partial v_{22}}{\partial y_2} = -M_\infty^2 U_0 \frac{\partial p_{21}}{\partial x_2}$$

$$U_0 \frac{\partial \rho_{22}}{\partial x_2} + v_{22} \frac{dR_0}{dy_2} = +M_\infty^2 R_0 U_0 \frac{\partial p_{21}}{\partial x_2} -$$

$$\frac{\partial A_1}{\partial t_2} \frac{dR_0}{dy_2} + A_1 \frac{\partial A_1}{\partial x_2} \left(U_0 \frac{d^2 R_0}{dy_2^2} - \frac{dR_0}{dy_2} \frac{dU_0}{dy_2}\right)$$

where the arbitrary function $A_2(t_2, x_2)$ must satisfy the condition $A_2 \to 0$ when $x_2 \to -\infty$ and $t_2 = \text{const}$.

We pass to the analysis of the boundary layer region III, where voscosity provides the predominant effect on the pattern of the flow field. In that region we have to set

$$t = \frac{L}{U_\infty}(t_0 + \varepsilon^2 t_3), \quad x = L(1 + \varepsilon^3 x_3), \quad y = \varepsilon^5 L y_3 \tag{1.15}$$

and represent expansions for gas parameters in the form

$$\begin{aligned} v_x &= U_\infty [\varepsilon u_{31}(t_3, x_3, y_3) + \varepsilon^2 u_{32}(t_3, x_3, y_3) + \ldots] \\ v_y &= U_\infty [\varepsilon^3 v_{31}(t_3, x_3, y_3) + \varepsilon^4 v_{32}(t_3, x_3, y_3) + \ldots] \\ \rho &= \rho_\infty [\rho_{31}(t_3, x_3, y_3) + \varepsilon \rho_{32}(t_3, x_3, y_3) + \ldots] \\ p &= p_\infty + \rho_\infty U_\infty^2 [\varepsilon^2 p_{31}(t_3, x_3, y_3) + \varepsilon^3 p_{32}(t_3, x_3, y_3) + \ldots] \end{aligned} \tag{1.16}$$

Comparison of formulas (1.1), (1.9), and (1.15) shows that $t_1 = t_2 = t_3$, $x_1 = x_2 = x_3$, but $y_1 \neq y_2 \neq y_3$. This is natural, since the length of all three regions in the direction of the oncoming stream is the same and the reading of time in these is carried out in the same manner. The transverse scales are, however, selected differently. As in region II, the scales of the two Cartesian coordinates in region III are dissimilar.

Expansions (1.16) can be simplified prior to their substitution into the Navier-Stokes equations. We assume that the plate in the stream is thermally insulated and that the Prandtl number is equal unity. The ratio T_W/T_∞ of wall temperature to that of the oncoming stream then satisfies Crocco's law [10]

$$\frac{T_W}{T_\infty} = 1 + \frac{\varkappa - 1}{2} M_\infty^2$$

where $\varkappa$ is the exponent of Poissons adiabatic curve, and furthermore the derivative $dR_0(0)/dy_2 = 0$. Hence we conclude that $\rho/\rho_\infty \to R_0(0)$ not only in the first but, also, in the second approximation when $x \to -\infty$. It is shown below that the condition of joining solutions for regions II and III yields the similar result: $\rho/\rho_\infty \to R_0(0)$ when $y_3 \to +\infty$. We therefore take $\rho_{31}(t_3, x_3, y_3) = R_0(0)$ and $\rho_{32}(t_3, x_3, y_3) = 0$ as the solution.

When the dependence $\lambda_1/\lambda_\infty = CT/T_\infty$ of the first viscosity coefficient on temperature is linear, the simplification of the system of Navier-Stokes equations yields for the unsteady boundary layer in an incompressible fluid the usual Prandtl equations

$$\frac{\partial u_{31}}{\partial x_3} + \frac{\partial v_{31}}{\partial y_3} = 0, \quad \frac{\partial p_{31}}{\partial y_3} = 0 \tag{1.17}$$

$$R_0(0)\left(\frac{\partial u_{31}}{\partial t_3} + u_{31}\frac{\partial u_{31}}{\partial x_3} + v_{31}\frac{\partial u_{31}}{\partial y_3}\right) = -\frac{\partial p_{31}}{\partial x_3} + \frac{C}{R_0(0)}\frac{\partial^2 u_{31}}{\partial y_3^2}$$

These equations are satisfied by the principal terms of expansions (1.16). The difference is in that it is not possible to take the perturbed pressure from the solution of the external flow problem. For the considered boundary layer of the plate $\partial p_{31}/\partial x_3 \neq 0$.

For second approximation functions we have

$$\frac{\partial u_{32}}{\partial x_3} + \frac{\partial v_{32}}{\partial y_3} = 0, \quad \frac{\partial p_{32}}{\partial y_3} = 0 \tag{1.18}$$

$$R_0(0)\left(\frac{\partial u_{32}}{\partial t_3} + u_{31}\frac{\partial u_{32}}{\partial x_3} + u_{32}\frac{\partial u_{31}}{\partial x_3} + v_{31}\frac{\partial u_{32}}{\partial y_3} + v_{32}\frac{\partial u_{31}}{\partial y_3}\right) =$$
$$-\frac{\partial p_{32}}{\partial x_3} + \frac{C}{R_0(0)}\frac{\partial^2 u_{32}}{\partial y_3^2}$$

which is nothing else but the linearized Prandtl equations for unsteady flows of an incompressible fluid. The remaining terms in the input system of Navier-Stokes equations affect only the derivation of higher approximations. The homogeneity of equations that constitute system (1.18) is related to that property.

2. Formulation of boundary value problems. To effect the joining of the considered asymptotic expansions it is necessary to determine the behavior of solution when approaching the upper and lower boundaries of region II from inside. Since $U_0(y_2) \to 1$ when $y_2 \to \infty$, formulas (1.14) yield

$$p_{22} - y_2\frac{\partial^2 A_1}{\partial x_2^2} \to p_{22}(t_2, x_2, 0) - \frac{\partial^2 A_1}{\partial x_2^2}\int_0^\infty \frac{M_\infty^2 - M_0^2}{M_\infty^2}\,dY_2 \tag{2.1}$$

$$v_{22} + (M_\infty^2 - 1)\,y_2\frac{\partial p_{21}}{\partial x_2} \to -\frac{\partial A_1}{\partial t_2} - \frac{\partial A_2}{\partial x_2}$$

$$u_{22} \to -p_{21}(t_2, x_2), \quad \rho_{22} \to M_\infty^2 p_{21}(t_2, x_2)$$

which are valid for any conditions at the plate. The behavior of solution in proximity of the lower boundary of region II depends on the thermal state that is maintained on its surface. As before, we assume that the plate is thermally insulated. Then, in addition to the previously noted equality $dR_0(0)/dy_2 = 0$, the relationships

$$\frac{d^2U_0(0)}{dy_2^2} = 0, \quad \frac{d^2R_0(0)}{\partial y_2^2} = (\varkappa - 1)\,M_\infty^2 N_{\Pr}\left[R_0(0)\frac{dU_0(0)}{dy_2}\right]^2$$

which follow from the Blasius solution [10] are valid on that surface. Taking these into account it is possible to show that for $y_2 = 0$ functions

$$p_{22} = p_{22}(t_2, x_2, 0) \tag{2.2}$$

$$v_{22} = -\left[R_0(0)\frac{dU_0(0)}{dy_2}\right]^{-1}\frac{\partial p_{21}}{\partial x_2} - \frac{\partial A_1}{\partial t_2} - A_1\frac{\partial A_1}{\partial x_2}\frac{dU_0(0)}{dy_2}$$

$$u_{22} = \frac{dU_0(0)}{dy_2} A_2 + M_\infty^2 \frac{dU_0(0)}{dy_2} p_{21}(t_2, x_2) \int_0^\infty \left\{ \frac{1}{M_0^2} - \left[Y_2 \frac{dM_0(0)}{dY_2} \right]^{-2} - \frac{1}{M_\infty^2} \right\} dY_2 \tag{2.2}$$

$$\rho_{22} = M_\infty^2 R_0(0) p_{21}(t_2, x_2) + \left\{ \frac{1}{2} A_1^2 + \frac{p_{21}(t_2, x_2)}{R_0(0)} \left[\frac{dU_0(0)}{dy_2} \right]^{-2} \right\} \frac{d^2 R_0(0)}{dy_2^2}$$

We use formulas (1.13) for the principal terms of expansions and the asymptotic formulas (2.1) for second approximation functions for obtaining the boundary conditions which must be satisfied for deriving the solution in region I. Since the external variable $y_1 \to 0$ when $y_2 \to \infty$, hence

$$p_{11}(t_1, x_1, 0) = p_{21}(t_2, x_2,) \quad v_{11}(t_1, x_1, 0) = -\frac{\partial A_1}{\partial x_1} \tag{2.3}$$

$$u_{11}(t_1, x_1, 0) = -p_{21}(t_2, x_2) \quad \rho_{11}(t_1, x_1, 0) = M_\infty^2 p_{21}(t_2, x_2)$$

Furthermore for pressure perturbations and for the transverse component of the velocity vector the relationships

$$p_{21}(t_1, x_1, 0) = p_{22}(t_2, x_2, 0) - \frac{\partial^2 A_1}{\partial x_2^2} \int_0^\infty \frac{M_\infty^2 - M_0^2}{M_\infty^2} dY_2 \tag{2.4}$$

$$v_{12}(t_1, x_1, 0) = -\frac{\partial A_1}{\partial t_2} - \frac{\partial A_2}{\partial x_2}$$

are valid in the second approximation.

Boundary condition for density perturbations and for the longitudinal component of the velocity vector cannot be obtained with the use of asymptotic formulas (2.1) when $y_1 = 0$. For this the terms of the third approximation solution for region II must be known. These are not considered in the present analysis.

Taking into consideration formulas (1.5) and (1.6) we conclude that boundary conditions for the principal terms of solution in region I can be expressed in terms of function $A_1(t_2, x_2) = A_1(t_1, x_1)$ when $y_1 = 0$. By virtue of formulas (1.7) and (1.8) boundary conditions for terms of the second approximation contain also functions $A_2(t_2, x_2) = A_2(t_1, x_1)$.

Let us effect the joining of expansions which represent the asymptotic form of solutions in regions II and III. Using formulas (1.13) and (2.2) we obtain the limit conditions which must be satisfied by the gas parameters within the boundary layer. When $y_2 \to 0$, the internal variable $y_3 \to \infty$ and the sought quantities

$$p_{31}(t_3, x_3) \to p_{21}(t_2, x_2) \tag{2.5}$$

$$u_{31} - \frac{dU_0(0)}{dy_2} y_3 \to \frac{dU_0(0)}{dy_2} A_1(t_2, x_2) + o\left(\frac{1}{y_3}\right)$$

The limit condition $\rho_{31} \to R_0(0)$ for density perturbation was used above as one of the reasons for selecting $\rho_{31}(t_3, x_3, y_3) = R_0(0)$ as the solution. The limit condition for the transverse component of the velocity vector is usually omitted. In the case considered here it can be represented thus

$$v_{31} - \frac{\partial A_1}{\partial x_2}\frac{dU_0(0)}{\partial y_2} y_3 \to -\left[R_0(0)\frac{dU_0(0)}{dy_2}\right]^{-1}\frac{\partial p_{21}}{\partial x_2} - \frac{\partial A_1}{\partial t_2} - A_1\frac{\partial A_1}{\partial x_2}\frac{dU_0(0)}{dy_2}$$

and is automatically satisfied when conditions (2.5) for pressure perturbations and for the transverse component of the velocity vector are satisfied. In fact, by solving the third of Eqs. (1.17) for function v_{31} we obtain

$$v_{31} = \left(\frac{\partial u_{31}}{\partial x_3}\right)^{-1}\left[-\frac{1}{R_0(0)}\frac{\partial p_{31}}{\partial x_3} + \frac{C}{R_0^2(0)}\frac{\partial^2 u_{31}}{\partial y_3^2} - \frac{\partial u_{31}}{\partial t_3} - u_{31}\frac{\partial u_{31}}{\partial x_3}\right]$$

Validity of the above statement can be proved by the asymptotic substitution of expressions for p_{31} and u_{31} in (2.5) into the last formula.

Limit conditions for second approximation functions are of the form

$$p_{32}(t_3, x_3) \to p_{22}(t_2, x_2, 0) \tag{2.6}$$

$$u_{32} \to \frac{dU_0(0)}{dy_2}\left\{A_2 + M_\infty^2 p_{21}(t_2, x_2)\int_0^\infty \left[\frac{1}{M_0^2} - \left[Y_2\frac{dM_0(0)}{dY_2}\right]^{-2} - \frac{1}{M_\infty^2}\right] dY_2\right\}$$

The condition $\rho_{32} \to 0$ has already been taken into account in the selection of solution $\rho_{32}(t_3, x_3, y_3) = 0$. Introducing into the second of equalities (2.2) the additional term to v_{22} which is proportional to y_2, for the transverse component of the velocity vector we obtain the limit condition

$$v_{32} \to -y_3\frac{dU_0(0)}{dy_2}\left\{\frac{\partial A_2}{\partial x_2} + M_\infty^2\frac{\partial p_{21}}{\partial x_2}\int_0^\infty \left[\frac{1}{M_0^2} - \left[Y_2\frac{dM_0(0)}{dY_2}\right]^{-2} - \frac{1}{M_\infty^2}\right] dY_2\right\}$$

which is automatically satisfied, as in the first approximation, when the conditions for pressure perturbations and for the longitudinal component of the velocity vector are satisfied. This can be proved by solving the third equation of system (1.18) for function v_{32} and substituting into the right-hand side of the obtained equation the asymptotic values of p_{31}, u_{31}, v_{31}, p_{32}, and u_{32}.

Note that in the process of joining solutions for regions II and III the term containing ρ_{22} in the expansion of density was altogether neglected. This is reasonable, since its contribution is proportional to ε^2 when $y_2 \to 0$, while it is sufficient to specify the gas density throughout the thin boundary layer with an accuracy to terms of order ε.

We use the conditions of joining for formulating the boundary value problems in region III for the first and second approximation functions. First, we effect the transformation

$$
\begin{aligned}
t_1 &= t_2 = t_3 = C^{1/4}\lambda^{-3/2}\,|M_\infty{}^2 - 1|^{-1/4}\,(T_W/T_\infty)\,t' \\
x_1 &= x_2 = x_3 = C^{3/8}\lambda^{-5/4}\,|M_\infty{}^2 - 1|^{-3/8}\,(T_W/T_\infty)^{3/2}\,x' \\
y_3 &= C^{5/8}\lambda^{-3/4}\,|M_\infty{}^2 - 1|^{-1/8}\,(T_W/T_\infty)^{3/2}\,y' \\
u_{31} + \varepsilon u_{32} &= C^{1/8}\lambda^{1/4}\,|M_\infty{}^2 - 1|^{-1/8}\,(T_W/T_\infty)^{1/2}\,(u_{31}' + \varepsilon u_{32}') \\
v_{31} + \varepsilon v_{32} &= C^{3/8}\lambda^{3/4}\,|M_\infty{}^2 - 1|^{1/8}\,(T_W/T_\infty)^{1/2}\,(v_{31}' + \varepsilon v_{32}') \\
p_{31} + \varepsilon p_{32} &= C^{1/4}\lambda^{1/2}\,|M_\infty{}^2 - 1|^{-1/4}\,(p_{31}' + \varepsilon p_{32}') \\
A_1 + \varepsilon A_2 &= C^{5/8}\varepsilon\lambda^{-3/4}\,|M_\infty{}^2 - 1|^{-1/8}\,(T_W/T_\infty)^{3/2}\,(A_1' + \varepsilon A_2')
\end{aligned}
\tag{2.7}
$$

where the constant $\lambda = 0.3321$ is determined by the equality

$$dU_0(0)/dy_2 = \lambda C^{-1/2}\,(T_W/T_\infty)^{-1}$$

and calculated using the Blasius solution for the unperturbed boundary layer. This transformation makes it possible to eliminate from the formulation of the problem the dependence of the principal terms of expansion on the rest $M_\infty - 1$ that is introduced by equality (1.5) or (1.6), and on the constants C and $R_0(0) = (T_W/T_\infty)^{-1}$.

In primed variables the system of Eqs. (1.17) for first approximation functions is of the canonical form

$$
\begin{aligned}
&\frac{\partial u_{31}'}{\partial x'} + \frac{\partial v_{31}'}{\partial y'} = 0, \quad \frac{\partial p_{31}'}{\partial y'} = 0 \\
&\frac{\partial u_{31}'}{\partial t'} + u_{31}'\frac{\partial u_{31}'}{\partial x'} + v_{31}'\frac{\partial u_{31}'}{\partial y'} = -\frac{\partial p_{31}'}{\partial x'} + \frac{\partial^2 u_{31}'}{\partial y'^2}
\end{aligned}
\tag{2.8}
$$

Boundary conditions for $y' = 0$ are obviously

$$u_{31}' = 0, \quad v_{31}' = 0 \tag{2.9}$$

The remaining boundary conditions are specified as limit conditions, namely, when $x' \to -\infty$ we have

$$u_{31}' \to y', \quad p_{31}' \to 0 \tag{2.10}$$

Moreover, on the basis of formulas (2.3) and (2.5) we conclude that when $y' \to \infty$

$$u_{31}' - y' \to A'(t', x') + o\left(\frac{1}{y'}\right) \tag{2.11}$$

and then one of the relationships

$$
\begin{aligned}
&p_{31}' = -\frac{\partial A_1'}{\partial x'}, \quad \text{if} \quad M_\infty > 1 \\
&p_{31}' = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\partial A_1'/\partial X'}{x' - X'}\,dX', \quad \text{if} \quad M_\infty < 1
\end{aligned}
\tag{2.12}
$$

is valid.

System (2.8) without the term $\partial u_{31}'/\partial t'$ in the left-hand side of the last of its equations was obtained in [5-9] in investigations of steady flows. The above considerations show that the formulation of the problem for an unsteady boundary layer is not unduly complicated because the first approximation equations that define the flow in regions I and II do not contain time derivatives.

Since neither the system of Eqs. (2.8) nor the boundary conditions (2.9)- (2.12) contain parameters Re, M_∞, $\varkappa$, C, λ, and T_W/T_∞, transformation (2.7) virtually defines the law of similarity for the principal terms of flow parameters in an unsteady boundary layer. Thus the same flow patterns may exist with various values of the indicated constants, and the properties of the flow are determined by the problem initial data. The similarity law for steady flows was initially established experimentally [3] and then given a theoretical basis in [5,7].

For second approximation functions system (1.18) consists of linearized Prandtl equations. Hence, in terms of primed variables it is also of the canonical form

$$\frac{\partial u_{32}'}{\partial x'}+\frac{\partial v_{32}'}{\partial y'}=0,\quad \frac{\partial p_{32}'}{\partial y'}=0 \tag{2.13}$$

$$\frac{\partial u_{32}'}{\partial t'}+u_{31}'\frac{\partial u_{32}'}{\partial x'}+u_{32}'\frac{\partial u_{31}'}{\partial x'}+v_{31}\frac{\partial u_{32}'}{\partial y'}+v_{32}'\frac{\partial u_{31}'}{\partial y'}=$$

$$-\frac{\partial p_{32}'}{\partial x'}+\frac{\partial^2 u_{32}'}{\partial y'^2}$$

which is free of any parameters. Any parameters are also absent in the boundary conditions $u_{32}'=0$ and $v_{32}'=0$ when $y'=0$, and in the limit conditions $u_{32}'\to 0$ and $p_{32}'\to 0$ when $x'\to-\infty$. The position with limit conditions when $y'\to\infty$ is, however, different. Before adducing these, it is expedient to calculate the integrals appearing in the asymptotic formulas (2.1) and (2.2). By the Blasius solution for the boundary layer of the plate the equalities

$$\int_0^\infty \frac{M_\infty^2-M_0^2}{M_\infty^2}\,dY_2=(2C)^{1/2}\frac{T_W}{T_\infty}\Delta_1,\quad \Delta_1=1.686$$

$$\int_0^\infty\left\{\frac{1}{M_0^2}-\left[Y_2\frac{dM_0(0)}{dY_2}\right]^{-2}-\frac{1}{M_\infty^2}\right\}dY=\overline{\int_0^\infty}\left(\frac{1}{R_0U_0^2}-1\right)dY_2=$$

$$(2C)^{1/2}\left[\left(\frac{T_W}{T_\infty}\right)^2\Delta_2-\frac{T_W}{T_\infty}\left(\frac{T_W}{T_\infty}-1\right)\Delta_1\right],\quad \Delta_2=-3.663$$

are valid [10]. In these formulas the angle superscript at the improper divergent integral denotes its finite part in the meaning of Adamard. Taking into consideration the conditions of joining (2.4) and (2.6) with $y'\to\infty$, we have as the result

$$u_{32}'\to A_2'+2^{1/2}C^{1/8}\lambda^{5/4}\,|M_\infty^2-1|^{-1/8}\left(\frac{T_W}{T_\infty}\right)^{1/2}\left(\Delta_2-\frac{T_W-T_\infty}{T_W}\Delta_1\right)p_{31}' \tag{2.14}$$

When $M_\infty>1$ the perturbed pressure is

$$p_{32}'=(2-M_\infty^2)\,C^{1/8}\lambda^{1/4}\,|M_\infty^2-1|^{-9/8}\left(\frac{T_W}{T_\infty}\right)^{1/2}\frac{\partial A_1'}{\partial t'}-\frac{\partial A_2'}{\partial x'}+ \tag{2.15}$$

$$2^{1/2}\Delta_1C^{1/8}\lambda^{5/4}\,|M_\infty^2-1|^{7/8}\left(\frac{T_W}{T_\infty}\right)^{-1/2}\frac{\partial^2 A_2'}{\partial x'^2}$$

For subsonic flows at $M_\infty<1$ the similar equality is considerably more complex. Recalling relationship (1.8) between pressure perturbation and the vertical component of the velocity vector at the lower boundary of region I, we obtain in this case

$$p_{32}' = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[2C^{1/8}\lambda^{1/4}|M_\infty^2 - 1|^{-1/8}\left(\frac{T_W}{T_\infty}\right)^{1/2}\frac{\partial A_1'}{\partial t'} + \frac{\partial A_2'}{\partial X'}\right]\frac{dX'}{x' - X'} + \tag{2.16}$$

$$2^{1/2}\Delta_1 C^{1/8}\lambda^{5/4}|M_\infty^2 - 1|^{7/8}\left(\frac{T_W}{T_\infty}\right)^{-1/2}\frac{\partial^2 A_1'}{\partial x'^2} +$$

$$\frac{1}{\pi}C^{1/8}\lambda^{1/4}|M_\infty^2 M_\infty^2 - 1|^{-9/8}\left(\frac{T_W}{T_\infty}\right)^{1/2}\int_0^\infty\int_{-\infty}^{+\infty}\frac{\partial^2 p_{11}'(t', X', Y')}{\partial t'\partial X'}\times$$

$$\left\{\ln\sqrt{b[(x' - X')^2 + Y'^2]} + \frac{1}{\pi}\left[\arccos\frac{-x' + X'}{\sqrt{(x' - X')^2 + Y'^2}}\times\right.\right.$$

$$\ln\sqrt{b[(x' - X')^2 + Y'^2]} + \arccos\frac{-x' - X'}{\sqrt{(x' + X')^2 + Y'^2}}\times$$

$$\left.\left.\ln\sqrt{b[(x' + X')^2 + Y'^2]}\right]\right\}dX'dY'$$

where the derivative $\partial^2 p_{11}'(t', X', Y')/\partial t'\partial X'$ and the constant b are defined by formulas

$$\frac{\partial^2 p_{11}'(t', X'Y')}{\partial t'\partial X'} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\partial^2 A_1'(t', X'')}{\partial t'\partial X''}\,\frac{(X' - X'')^2 - Y'^2}{[(X' - X'')^2 + Y'^2]^2}dX''$$

$$b = C^{3/4}\lambda^{-5/4}|M_\infty^2 - 1|^{-3/4}\left(\frac{T_W}{T_\infty}\right)^3$$

Derivation of the limit condition (2.14) with its associated equalities (2.15) and (2.16) for the perturbed pressure completes the formulation of the boundary value problem for second approximation functions that saisfy the system of linear equations (2.13). Here the correction terms substantially depend on constants M_∞, C, λ, and T_W/T_∞.

3. The linear approximation. Let us consider a supersonic oncoming stream i.e. at $M_\infty > 1$. When expanding gas parameters into asymptotic sequencies we shall retain only the principal terms and neglect terms of second order of smallness. We then have to solve in region III the system of nonlinear equations (2.8) with boundary conditions (2.9)-(2.11), and to determine pressure by the first of formulas (2.12). The stated problem is homogeneous and has evidently the trivial solution $u_{31} = y$, $v_{31} = 0$, and $p_{31} = 0$ which corresponds to the continuation of Blasius solution in unaltered form throughout the considered region. Here and in what follows we omit the primes at independent variables and unknown functions. The existence of one more steady solution in the linear theory framework was established in [4]. This is of fundamental importance, since it shows that the Blasius solution can bifurcate at the boundary of the boundary layer when $x \to -\infty$. The bifurcation of solution in the steady case makes possible the introduction of the hypothesis of the existence of a whole class of gas motions with parameters that depend on time and continuously adjoin the unperturbed boundary layer. To test that hypothesis we write the solution as

$$u_{31} = y - ae^{\omega t + kx}df/dy, \quad v_{31} = ake^{\omega t + kx}f(y) \tag{3.1}$$
$$p_{31} = ae^{\omega t + kx}$$

Since the first and second equations of system (2.8) are linear, they indentically satisfy these solutions. As regards the last equation, its linearization with respect to the amplitude a of perturbations yields for function $f(y)$ the third order ordinary differential eqaution

$$\frac{d^3f}{dy^3} - (\omega + ky)\frac{df}{dy} + kf + k = 0 \tag{3.2}$$

The constants ω and k in formulas (3.1) are assumed to be complex, hence amplitude a and $f(y)$ are also complex. Thus

$$\omega = \omega_1 + i\omega_2, \ k = k_1 + ik_2, \ a = a_1 + ia_2, \ f(y) = f_1(y) + if_2(y)$$

which implies that Eq. (3.2) defines a complex function of a real variable. The boundary conditions for $y = 0$ are derived from equality (2.9) and are of the form

$$f = df/dy = 0 \tag{3.3}$$

To satisfy the limit conditions (2.10) when $x \to -\infty$ it is sufficient to set $k_1 > 0$. The limit condition (2.11) when $y \to \infty$ yields

$$\frac{df}{dy} \to \frac{1}{k} \tag{3.4}$$

The boundary value problem (3.3), (3.4) has been thus formulated for the differential equation (3.2). Since the equation is of the third order and the problem contains only three conditions, it would appear at first sight that its solution exists for any ω and k. But this is not so, since one of the three linearly independent solutions of Eq. (3.2) has an unbounded derivative when $y \to \infty$. Hence condition (3.4), while on the one hand stipulates the rejection of that solution, yields on the other hand an equality which must be satisfied by the multiplicative constant in the solution with a bounded derivative. This clearly shows that the boundary value problem (3.3), (3.4) is a problem in eigenvalues.

To solve the formulated problem we use the following method. We differentiate Eq. (3.2) and apply the transformation $z = \omega/k^{2/3} + k^{1/3}y$ to the independent variable. As the result we obtain the fourth order differential equation

$$\frac{d^4f}{dz^4} - z\frac{d^2f}{dz^2} = 0 \tag{3.5}$$

with boundary conditions

$$f = 0, \quad \frac{df}{dz} = 0, \quad \frac{d^3f}{dz^3} = -1 \quad \text{for} \quad z = \frac{\omega}{k^{2/3}} \tag{3.6}$$

$$df/dz = k^{-4/3} \quad \text{for} \quad |z| \to \infty$$

Since $k_1 = \operatorname{Re} k > 0$, the inequalities $-\pi/6 < \arg k^{1/3} < \pi/6$ are valid, hence $-\pi/6 < \arg z < \pi/6$ when $y \to \infty$. It will be readily seen that Eq.(3.5) represents the canonical form of the Airy equation for the second derivative d^2f/dz^2. Its solution which satisfies condition $|df/dz| < \infty$ can be presented in the form

$$d^2f/dz^2 = c_0 \operatorname{Ai}(z)$$

where $\operatorname{Ai}(z)$ is the Airy function of the complex variable, and c_0 is an arbitrary constant. The condition for the third derivative at point $z = \omega/k^{2/3}$ determines that constant

$$c_0 = -\left[\frac{d\,\mathrm{Ai}\,(\omega/k^{2/3})}{dz}\right]^{-1}$$

and the remaining conditions at that point make it possible to write the solution as

$$f = -\left[\frac{d\,\mathrm{Ai}\,(\omega/k^{2/3})}{dz}\right]^{-1} \int\limits_{\omega/k^{2/3}}^{z} \left[\int\limits_{\omega/k^{2/3}}^{z} \mathrm{Ai}\,(Z')\,dZ'\right] dZ \tag{3.7}$$

Finally, the last of conditions (3.6) which restricts the order of solution increase at the infinitely distant point yields the variance relationship

$$-\frac{d\,\mathrm{Ai}\,(\omega/k^{2/3})}{dz}\left[\int\limits_{\omega/k^{2/3}}^{\infty} \mathrm{Ai}\,(Z)\,dZ\right]^{-1} = k^{4/3} \tag{3.8}$$

which links the exponents ω and k. Values of these exponents, which satisfy formula (3.8) are eigenvalues in the solution of the input boundary value problem.

When $\omega = 0$, the dependence on time in the linear solution (3.1) vanishes, and the flow of gas in the boundary layer is steady. As established earlier in [4], in this case $k\,(0) = k_1{}^* = [3^{7/4}\,\Gamma\,(^2/_3)/2\pi]^{3/4}$, $k_2 = 0$. To determine the general function $k\,(\omega)$ we use the expansion of Airy's function into series [11]

$$\mathrm{Ai} = \frac{\pi^{1/2}}{3^{2/3}}\left[\sum_{m=0}^{\infty} \frac{z^{3m}}{m!\,3^{2m}\Gamma\,(m+{}^2/_3)} - \frac{z}{3^{1/3}}\sum_{m=0}^{\infty} \frac{z^{3m}}{m!\,3^{2m}\Gamma\,(m+{}^4/_3)}\right] \tag{3.9}$$

whose radius of convergence is infinite. From this for small $|\,\omega\,|$ we have

$$\omega = [k^{2/3}\,\mathrm{Ai}\,(0)]^{-1}\left[\frac{d\,\mathrm{Ai}\,(0)}{dz} + k^{4/3}\int\limits_{0}^{\infty} \mathrm{Ai}\,(Z)\,dZ\right] =$$

$$3^{-1/3}\,\Gamma\,(^2/_3)\,k^{2/3} - 2^{-1}3^{5/6}\pi^{-1}\Gamma^2\,(^2/_3)\,k^{-2/3}$$

It is obvious that complex values of k obtain only when ω is complex. If ω is made real, then k is also real. This conclusion remains valid also for considerable $|\,\omega\,|$.

The complete solution of the transcendental equation was numerically obtained on a computer. Denoting its left-hand side by Φ we have

$$\Phi\,(w) = k^{4/3}, \quad w = \omega/k^{2/3}$$

We select an arbitrary $\omega = \omega_j$ and determine the corresponding $k = k_j(\omega_j)$ using Newton's method. Let $k_j{}^i$ be the i-th approximation of that unknown quantity, then $w_j{}^i = \omega_j/(k_j{}^i)^{2/3}$ and the correction

$$\Delta k_j{}^i = [k_j{}^i - \Phi\,(w_j{}^i)]\left\{\frac{4}{3}k_j{}^i\left[1 + \frac{1}{2}\frac{\omega_j}{(k_j{}^i)^2}\frac{d\Phi}{dw}\right\}\right]^{-1} \tag{3.10}$$

Owing to the analiticity of function $\Phi\,(w)$ its derivative $d\Phi/dz$ is independent of direction in the complex plane, which makes formula (3.10) very convenient for calculations. Taylor expansion (3.9) for Airy's function, whose argument variation was limited by inequality $|\,z\,| < 7$, was used in calculations. The quantity k_{j-1} which corresponds to the preceding ω_{j-1} was specified for the initial approximation $k_j{}^0$.

With a step $|\Delta\omega| = 0.1$ formula (3.10) makes it possible to determine k to within six digits after two to three iterations. For considerable values of $|z| \to \infty$ it is necessary to use an asymptotic expansion of Airy's function instead of Taylor series [11].

Curves calculated for $k(\omega)$ are shown in Fig. 1 solid lines relate to k_1 and dash lines to k_2. The quantity $\omega_2 = \mathrm{Im}\,\omega$ is taken as the independent variable and $\omega_1 = \mathrm{Re}\,\omega$ appears as a parameter whose numerical values denote the related curves. It will be seen that all curves for $\omega_1 > 0$ lie on one side of the curve for $\omega_1 = 0$. Functions $k_1(\omega_2)$ and $k_2(\omega_2)$ are, respectively, even and odd and independent of the value of ω_1. Both the real and the imaginary parts of k monotonically increase with increasing ω_1 and ω_2.

Curves with $\omega_1 = 0$ are limit curves, since then solution (3.1) is purely oscillatory in time. Oscillations in time are accompanied by oscillations in space but, since $k_1 > 0$, the latter are attenuated in conformity with the exponential law when $x \to -\infty$. We therefore conclude that the supersonic boundary layer on the plate surface can be steady up to a certain point beyond which lies a region where gas parameters are subject to periodic variation with constant amplitudes at each place.

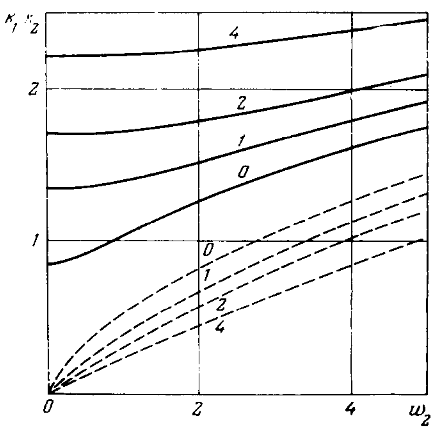

Fig. 1

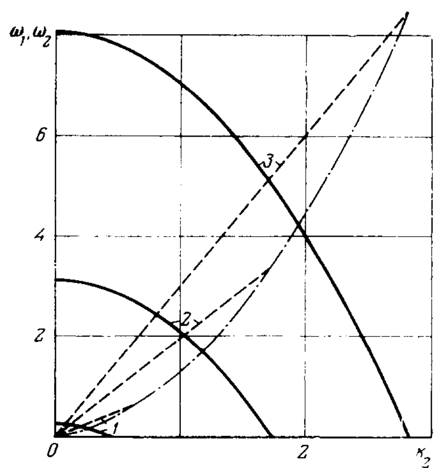

Fig. 2

A property of the linear solution is that when $\omega_1 > 0$ the curves show the increasing amplitude of oscillations. To avoid an exponential increase of amplitude when $t \to \infty$ (and when $x \to \infty$) it is necessary to integrate the system of Eqs. (2.8) without recourse to its linearization.

In addition to the described program another was established for calculating the dependence $\omega(k)$. The results obtained by that program are shown in Fig. 2, where solid and dash lines relate, respectively, to ω_1 and ω_2. The quantity k_2 was taken as the independent variable, and values of parameter k_1 are indicated at their related curves. The quantity ω_1 drops fairly rapidly to zero with increasing k_2 and $k_1 =$ const. From this we determine the range of variation of the independent variable in the derivation of functions $\omega_2(k_2)$. Values of $\omega_1 < 0$ are not considered here, since they relate to oscillations which dampen in time.

We shall discuss in conclusion the link between the solutions considered here and the nonlinear theory of viscous flow stability. So far the investigated phenomena were assumed to take place under conditions of free interaction between an unsteady boundary layer and an external potential flow. However the introduction of time as one of the arguments representing parameters of free interaction in the dependence, makes in itself possible to treat these from a different point of view. Thus in a particular case it is possible to formulate the problem of boundary layer stability in the same terms as the problem of development of perturbations in that layer, which lead to the formation of a self-induced pressure gradient.

In the nonlinear theory of viscous flow stability, which was the subject of recent surveys [12, 13], a considerable part is played by the concept of the critical layer. Its position, i.e. distance from the wall, is determined by the condition that the phase velocity of perturbation propagation is close to the velocity of the main stream specified by function $U_0(y_2)$. A thorough analysis of the boundary layer structure appears in [14, 15], where it is based on essentially nonlinear equations, although viscous stresses are taken there into account only for clarifying some details of the velocity field. The critical layer is originated when the perturbation amplitude reaches the threshold below which the linear theory of stability is valid.

Let us now assume that the velocity of wave propagation is close to zero. The critical layer then reaches the bottom of flow and merges with the boundary layer region III. Since perturbations in it are comparatively large, they must result in a self-induced pressure gradient. The latter is automatically attuned to the velocity field structure with the stream filament thickness varying with time. The stability problem is, thus, formulated similarly to that of free interaction of an unsteady boundary layer.

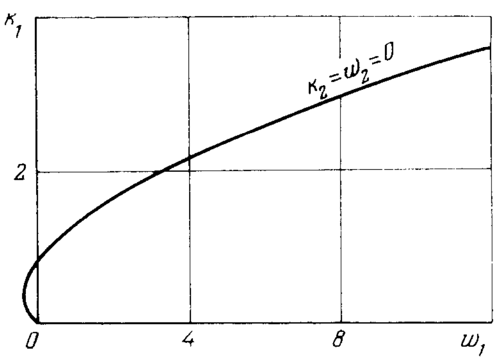

Fig. 3

Let us consider solution (3.1) with $\omega_2 = k_2 = 0$ using the described approach. The behavior of the dash curves in Fig. 1 immediately shows that in spite of the specification of a pair of numbers out of four which form the exponents ω and k, there exists a complete set of values for k_1 which define the dependence $k_1(\omega_1)$. When $\omega_2 = k_2 = 0$ solution (3.1) may be treated as a traveling wave. It is such perturbations that are considered in the theory of the critical layer [12-15]. Formula (3.7) by which function $f(y)$ is introduced then contains an integral of real quantities. We assume that the traveling wave is generated at instant $t = 0$ by the divergence of gas parameters that are distributed according to the law

$$u_{31} = y - ae^{k_1 x}\frac{df}{dy}, \qquad v_{31} = ake^{k_1 x} f(y), \qquad p_{31} = ae^{k_1 x} \tag{3.11}$$

The pattern of perturbed motion at $t > 0$ depends on number k_1. Values of exponent ω_1 are obtained from the curve in Fig. 3. When $k_1 = k_1^* = [3^{7/4}\,\Gamma(2/3)/2\pi]^{3/4}$, we have $\omega_1 = 0$. The initial data (3.11) represent the exact steady solution of the linear problem derived in [4]. Such wave remains stationary at all instants of time.

When $k_1 > k_1^*$, the exponent $\omega_1 > 0$, and the wave runs against the basic flow in the initial boundary layer. The propagation of perturbations upstream is in complete accord with the fundamental ideas of the theory of free interaction, but was not, so far, considered in problems of motion stability of a viscous fluid. The origin of this phenomenon is explained by the selection of a fairly high pressure gradient at the initial instant of time.

Finally, let $k_1 < k_1^*$ and consequently $\omega_1 < 0$. Under these conditions the wave travels downstream, as is usually assumed in the nonlinear theory of viscous flow stability. Thus the direction of wave propagation is determined by the pressure gradient in initial data. The comparatively small increase of excess pressure along the x-axis at $t = 0$ is incapable of forcing the penetration of perturbations upstream, while the basic flow in the initial boundary layer carries these downstream. When $k_1 \to 0$, $\omega_1 \to 0$ too, and formulas (3.1) are transformed into the trivial solution $u_{31} = y$, $v_{31} = 0$, and $p_{31} = 0$ for the Blasius boundary layer. The curve represented in Fig. 3 connects both points $\omega_1 = 0, k_1 = k_1^*$ and $w_1 = 0, k_1 = 0$ which determine the bifurcation of the linear solution when the gas flow is steady. At the second of these points the velocity of the wave propagating downstream proves to be infinite.

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